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NONSYMMETRIC RESOURCE NETWORKS. THE STUDY OF LIMIT STATES

Oleg P. Kuznetsov, Ludmila Yu. Zhilyakova

V.A. Trapeznikov Institute of Control Sciences, Moscow, Russia

Corresponding author:
Oleg P. Kuznetsov
V.A. Trapeznikov Institute of Control Sciences
65 Profsoyuznaya St., 117997 Moscow, Russia
phone: +74 95 334-76-39
e-mail: olkuznes@ipu.rssi.ru

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Abstract

A network model called a resource network and represented by an oriented weighted graph with loops is considered. In the bidirectional resource network: i) any two vertices are either not adjacent or connected by a pair of oppositely directed edges; ii) resources are assigned to vertices, which have unlimited volumes; the weights of arcs indicate their capacities. The total resource is constant, while resources at the vertices are reallocated according to certain rules in discrete time. The limit states of networks with arbitrary initial distribution of resource are analyzed. The threshold value T is proved to exist: when the total resource value is less than T the network with loops corresponds to a regular Markov chain; when the total resource value exceeds T the Markov property does not hold. The classification of vertices, depending on their ability to accumulate resources is given. The vertices capable to accumulate the amount of resource, surpassing their total output capacity was is were called the potential attractors. The criterion of attractiveness of vertices is formulated. The formulae of resource value at every vertex in limit state expressing its dependence on limit probabilities of corresponding Markov chain, the total resource value and total output capacity are derived.

Keywords

resource network, state of the network, oriented weighted graph, criterion of attractiveness of vertices.

Introduction

The resource network represented in [1] is a new dynamic model, in which the resource flows in networks of special kind are simulated. A number of applied problems related to flows in networks cannot be formalized within the frames of the classical Ford-Fulkerson model [2] and its dynamic modifications [3]. Among them we can name a management of resource allocation in virtual networks [4], modeling of distribution of chemical substances and passive biological objects in an aqueous medium [5], etc.

The suggested network is a bidirectional graph: adjacent vertices are connected by a pair of oppositely directed arcs. The weights of arcs denote their capacities (abilities to transfer resources). The resources are exchanged between vertices under certain rules with the total resource being constant. As opposed to the classical Ford-Fulkerson flow model, in which resources flow from sources to sinks and are contained at arcs, in resource network resources are contained at vertices and there is no sources, sinks, and general flow direction.

In [1] the complete homogeneous (with identical arc capacities) networks without loops were considered. In [5, 6] the complete heterogeneous networks with loops and their limit states were analyzed. In this paper we continue analyzing of the properties of nonsymmetric resource networks and their limit states.

The vertices transfer the resource along all output arcs following one of two rules depending on the amount of resource. If at the time t this amount is sufficient, every arc conducts the resource equal its capacity (Rule 1); the surplus remains at the vertex. Otherwise the vertex transfers all its resource along output arcs proportionally their capacities (Rule 2).

In [7] the existence and uniqueness of the threshold value T are proved: if the total resource value (W) does not exceed T all the vertices since some time t' transfer resource following Rule 2; if the total resource value is more than T, there exist at least one vertex functioning according to Rule 1. The formulae for coordinates of limit state vector Q^* with $W \leq T$ are derived. The stochastic matrix R', corresponding to the capacity matrix R is defined. It is shown that the limit of its powers $(R')^\infty$ is the matrix of limit probabilities of Markov chain and consists of n rows Q^{1*}, where Q^{1*} is row vector of limit distribution of unit resource. The limit state vector Q^* for $W \leq T$ is shown to be independent on initial distribution of resource, and moreover Q^* is an eigenvector of matrixes R' and $(R')^\infty$ with an eigenvalue $\lambda = 1$.

This paper studies the behavior of a network with $W > T$ and is devoted to finding the formulae for threshold value T and coordinates of the limit state vector Q^* and to generalization of the results obtained for different values of the resource.

Basic definitions

A *resource network* is a directed graph whose vertices v_i are assigned nonnegative numbers $q_i(t)$ (called *resources*) varying in discrete time t and whose arcs $(v_i,\ v_j)$ are assigned time-invariant positive numbers r_{ij} (called *capacities*); n is a number of vertices.

The *state of the network* at time t is defined by the vector $Q(t) = (q_1(t), \ldots, q_2(t), \ldots, q_n(t))$. At each time, resources are transferred from the vertices along the output arcs with the resource amounts depending on the arc capacities. The rules for resource transfer satisfy the following conditions:

(i) The network is closed; i.e., no resources are supplied from the outside and leak outside.

(ii) A resource amount sent out of a vertex is subtracted from its total resource, while a resource amount arriving at a vertex is added to its total resource; i.e., the total resource W is conserved:

$$\forall t \sum_{i=1}^{n} q_i(t) = W.$$

A state $Q(t)$ is called *stable* if $Q(t) = Q\ (t+1) = Q(t+2) = Q(t+3) = \ldots$.

A state $Q^* = (q_1^*, q_2^*, \ldots, q_n^*)$ is said to be *asymptotically reachable* from the state $Q(0)$ if for any $\varepsilon > 0$ there exists t_ε such that $|q_i^* - q_i(t)| < \varepsilon$, $i = 1, 2, \ldots, n$ for all $t > t_\varepsilon$.

A network state is called a *limit state* if it is either stable or asymptotically reachable.

A pair of arcs $< (v_i, v_j), (v_j, v_i) >$ is called a *bidirectional pair*. A network whose vertices are connected only by bidirectional pairs is called a *bidirectional network*.

A network is said to be *homogeneous* if all the capacities are identical (let them be equal to r). Otherwise it is called a *heterogeneous* network.

The capacity matrix of a network is defined as $R = ||r_{ij}||_{n\times n}$. The properties of R ensue directly from its definition:

(i) R is a nonnegative matrix: $\forall ij\ r_{ij} \geq 0$;

(ii) $\forall i\ r_{ii} > 0$;

(iii) $\forall ij\ (r_{ij} > 0 \Leftrightarrow r_{ji} > 0)$.

If the network is complete, R is a positive matrix.

The total capacity r_{sum} of a network is defined as the sum of the capacities of all its arcs:

$$r_{sum} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij}.$$

Let the total capacity $\sum\limits_{j=1}^{n} r_{ji}$ of all input arcs of the i-th vertex be denoted by r_i^{in} and called *total input capacity*; the total output capacity $\sum\limits_{j=1}^{n} r_{ij}$ be denoted by r_i^{out} and called *total output capacity*. The capacity of a loop is included in both sums.

So each vertex can be characterized by a pair (r_i^{in}, r_i^{out}). Thus the entire network is characterized by a tuple:

$$\rho = \left\{ \left(r_1^{in}; r_1^{out}\right), \left(r_2^{in}; r_2^{out}\right), \ldots \left(r_n^{in}; r_n^{out}\right) \right\}. \quad (1)$$

The rules for distribution of resource. At the time t the resource amount transferred by the vertex v_i into vertex v_m is:

$$r_{im} \text{ if } q_i(t) > r_i^{out} \quad \text{(Rule 1);}$$

$$\frac{r_{im}}{r_i^{out}} q_i(t) \quad \text{otherwise (Rule 2).}$$

Let us consider *nonsymmetric* networks – networks having vertices with different input and output capacities: $\left| r_i^{in} - r_i^{out} \right| > 0$. For arbitrary vertex v_i this difference is designated as Δr_i: $\Delta r_i = r_i^{in} - r_i^{out}$.

The vertices in a nonsymmetric network are divided into three classes:

(i) *vertices-receivers*, for which $\Delta r_i > 0$;

(ii) *vertices-sources*, $\Delta r_i < 0$;

(iii) *neutral vertices*, $\Delta r_i = 0$.

The path from the neutral vertex to the source that contains no receivers, is called *non-positive path*.

The set of vertices with a resource $q_i(t)$ not exceeding r_i^{out}, is called a zone $Z^-(t)$. The vertices from

$Z^-(t)$ are functioning according to the Rule 2. $Z^+(t)$ is a set of vertices, having the resource greater than their output capacity; they operate according to the Rule 1. For the limit state Q^* let us denote these zones as Z^{-*} and Z^{+*}.

T is a threshold value: if $W \le T$ all the vertices since some time t' pass into $Z^-(t)$; if $W > T$, since some time t'' zone $Z^+(t)$ is nonempty [7].

In [7] is proved that if $W \le T$ the process of resource distribution converges for any connected bidirectional network with loops and any initial distribution $Q(0)$.

The stochastic matrix

$$R' = \begin{pmatrix} \frac{r_{11}}{r_1^{out}} & \frac{r_{12}}{r_1^{out}} & \dots & \frac{r_{1n}}{r_1^{out}} \\ \dots & \dots & \dots & \dots \\ \frac{r_{n1}}{r_n^{out}} & \frac{r_{n2}}{r_n^{out}} & \dots & \frac{r_{nn}}{r_n^{out}} \end{pmatrix},$$

obtained from the capacity matrix R by normalization of rows has the following properties:

1. The limit state vector $Q^* = (q_1^*, q_2^*, ..., q_n^*)$ for any $W \le T$ is its left eigenvector corresponding to eigenvalue $\lambda_1 = 1 : Q^* = Q^* \cdot R'$.

2. For any matrix R', corresponding to a capacity matrix of the network with loops there exists $\lim\limits_{n\to\infty} (R')^n = (R')^\infty$, and the vector Q^* is also its left eigenvector corresponding to eigenvalue $\lambda_1 = 1$: $Q^* = Q^* \cdot (R')^\infty$.

3. $$(R')^\infty = \begin{pmatrix} Q^{1*} \\ Q^{1*} \\ \dots \\ Q^{1*} \end{pmatrix} = \begin{pmatrix} q_1^{1*} & q_2^{1*} & \dots & q_n^{1*} \\ q_1^{1*} & q_2^{1*} & \dots & q_n^{1*} \\ \dots & \dots & \dots & \dots \\ q_1^{1*} & q_2^{1*} & \dots & q_n^{1*} \end{pmatrix},$$

where Q^{1*} is vector of limit distribution of resource $W = 1$ [8, 9].

For any $W \le T$ the limit state vector is proportional to Q^{1*}:

$$Q^* = Q^{1*} \cdot W. \tag{2}$$

Vertices, belonging to Z^{+*} for $W > T$ are called *attractors*. Vertices, able to reach Z^{+*} from some initial state, are called the *potential attractors*. In [7] was proved, that a vertex is a potential attractor if and only if with $W = T$ it has in limit state the resource value equal to its output capacity. If such a vertex is unique it is the vertex-receiver. If there are several potential attractors, among they can be both receivers and neutral vertices without non-positive paths.

Let us denote the limit state vector for $W = T$ as $\widetilde{Q} = (\widetilde{q}_1, \tilde{q}_2, ..., \tilde{q}_n)$. Then the criterion of attractiveness of a vertex has the form: $\widetilde{q}_i = r_i^{out}$.

The collection of networks corresponding to one stochastic matrix

Every capacity matrix R corresponds to a unique matrix R':

$$R = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \dots & \dots & \dots & \dots \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{pmatrix},$$

$$R' = \begin{pmatrix} \frac{r_{11}}{r_1^{out}} & \frac{r_{12}}{r_1^{out}} & \dots & \frac{r_{1n}}{r_1^{out}} \\ \dots & \dots & \dots & \dots \\ \frac{r_{n1}}{r_n^{out}} & \frac{r_{n2}}{r_n^{out}} & \dots & \frac{r_{nn}}{r_n^{out}} \end{pmatrix}.$$

On the other hand, a stochastic matrix R' corresponds to an infinite set of capacity matrices R, since the proportional change of rows in R is invariant under R'.

For example, matrices R_1 and R_2 correspond to the same stochastic matrix R':

$$R_1 = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \\ 100 & 50 & 10 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 98 & 98 & 98 \\ 15 & 30 & 45 \\ 10 & 5 & 1 \end{pmatrix},$$

$$R' = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{5}{8} & \frac{5}{16} & \frac{1}{16} \end{pmatrix}.$$

Let us pay heed that the proportional change in the i-th row of R corresponds to the change in the capacity of *all output arcs* of the i-th vertex.

We investigate the limit distribution of resource for different capacity matrices, corresponding to the same stochastic matrix R', and analyze the influence of changes of total capacity on the threshold value T.

At first, let us give examples of the networks with different capacity matrices, corresponding to one stochastic matrix. These examples illustrate how (i) the limit state, (ii) the threshold value T, and (iii) the property of attractiveness depend on changes of the capacity matrix R, invariant under R'.

Example 1. Capacity matrix 3×3; total resource $W = 1$; the initial state: $Q(0) = (1, 0, 0)$.

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 5 \\ 4 & 4 & 4 \end{pmatrix}.$$

The corresponding stochastic matrix:

$$R' = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{10} & \frac{4}{10} & \frac{5}{10} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Table 1
Protocol of resource distribution[1].

t	v_1	v_2	v_3
0	1.000	0.000	0.000
1	0.333	0.333	0.333
2	0.256	0.356	0.389
3	0.250	0.357	0.393
4	0.250	0.357	0.393

The resulting limit state allows to construct a matrix $(R')^\infty$, consisting of three identical rows Q^{1*} ($Q^{1*} = (0.250, 0.357, 0.393)$):

$$(R')^\infty = \begin{pmatrix} 0.25 & 0.357 & 0.393 \\ 0.25 & 0.357 & 0.393 \\ 0.25 & 0.357 & 0.393 \end{pmatrix}.$$

The tuple ρ (see (1)) for this network is: $\rho = \{(6,3),\ (9,10),\ (10,12)\}$. The network has one receiver (the only potential attractor) and two sources. From the formula $Q^* = Q^{1*} \cdot W$, derived in [7] for $W \leq T$, follows that with increasing of W the resources in the vertices will grow proportionally, until W reaches the threshold value T. Further resources for all non-attractors become stable, and the accumulation occurs only in the attractor. In [7] is proved, that the threshold value T is reached when at least one of the vertices in the limit state receives a resource equal to its output capacity. For given network $T \approx 12$.

Example 2. The same capacity matrix $R_{3\times3}$; $Q(0) = (12, 0, 0)$.

Table 2
Protocol of resource distribution.

t	v_1	v_2	v_3
0	12.000	0.000	0.000
1	10.000	1.000	1.000
2	8.433	1.733	1.833
3	7.218	2.304	2.478
4	6.274	2.748	2.978
...			
38	3.001	4.285	4.714
39	3.000	4.286	4.714
40	3.000	4.286	4.714

As can be seen from the protocol, a resource of the receiver in the final distribution has reached the value $r_1^{out} = 3$.

With further increase of resource the receiver starts functioning according to the Rule 1, i.e. transfers the full capacity in each output arc accumulating a surplus. For example, for the total resource $W = 100$ in this network, the limit state (up to three decimal characters) will be: $Q^* = (91.000,\ 4.286,\ 4.714)$.

Thus, the threshold value of resource T is the sum: $T \approx 3 + 4.286 + 4.714 = 12$.

The increase of output capacity of sources

Now let us investigate the changes of the limit state vector with a proportional increase in all capacities of one of the sources.

Example 3. Total recourse $W = 1$. The third row of capacity matrix from Example 1 is multiplied by five:

$$R = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 5 \\ 20 & 20 & 20 \end{pmatrix}.$$

This matrix corresponds to the same stochastic matrix.

The protocol of distribution for total resource $W = 1$ is the same as the protocol of Example 1:

Table 3
Protocol of resource distribution.

t	v_1	v_2	v_3
0	1.000	0.000	0.000
1	0.333	0.333	0.333
2	0.256	0.356	0.389
3	0.250	0.357	0.393
4	0.250	0.357	0.393

It is easy to see that with the proportional increase of output capacities of the sources i) the vector of limit state Q^{1*} corresponding to $W = 1$ remains unchanged, and ii) the threshold value T remains constant, and if it is exceeded the resources at all non-attractors in the limit state do not change for any $W > T$.

The increase of output capacity of vertex-receiver

The unrestricted increase of output capacity of the receiver will cause a change of its status, i.e. from some value of output capacity it begins to send resource more than receive. For the beginning let us consider the increase of output capacity without changing of status. As a basis we take the matrix from Example 1.

[1]The finiteness of the table does not imply the finiteness of the process. Limit state is being reached asymptotically. Further in the subsequent rows the first three digits after the decimal point are the same.

Example 4. The capacity matrix $R_{3\times3}$; the total resource $W = 1$. $Q(0) = (1,\ 0,\ 0)$.

The first row of matrix from the Example 3 is multiplied by two:

$$R = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 4 & 5 \\ 4 & 4 & 4 \end{pmatrix}.$$

The tuple ρ is: $\rho = \{(7,6),(10,10),(11,12)\}$. The first vertex remained a receiver and the source and the neutral vertex reversed.

Table 4
Protocol of resource distribution.

t	v_1	v_2	v_3
0	1.000	0.000	0.000
1	0.333	0.333	0.333
2	0.256	0.356	0.389
3	0.250	0.357	0.393
4	0.250	0.357	0.393

Apparently, this protocol is completely coincided with protocols of the Examples 1 and 3.

The total output capacity of the receiver is doubled. As experiments show, at the same time the threshold value T and resource values on which a stabilization of the remaining vertices occurred are doubled too.

Example 5. The matrix R from Example 4; the total resource $W = 24 \approx T$. $Q(0) = (24,\ 0,\ 0)$.

Table 5
Protocol of resource distribution.

t	v_1	v_2	v_3
0	24.000	0.000	0.000
1	20.000	2.000	2.000
2	16.867	3.467	3.667
3	14.436	4.609	4.956
4	12.548	5.495	5.956
...			
46	6.000	8.572	9.428
47	6.000	8.572	9.428

$$T = 6 + 8.572 + 9.428 = 24.$$

The threshold value T is doubled. In addition, for each vertex from Example 2 holds: $3 \cdot 2 = 6$; $4.286 \cdot 2 = 8.572$; $4.714 \cdot 2 = 9.428$, i.e. all threshold values of vertices are doubled.

The replacement of a receiver

Let us consider again the matrix of Example 1 and increase the output capacity of receiver tripled.

Example 6. The replacement of the receiver. The total resource $W = 1$. $Q(0) = (1,\ 0,\ 0)$.

$$R = \begin{pmatrix} 3 & 3 & 3 \\ 1 & 4 & 5 \\ 4 & 4 & 4 \end{pmatrix}.$$

For such a configuration of the network the tuple ρ is: $\rho = \{(8,9),\ (11,10),\ (12,12)\}$.

All the vertices have changed their status. The first one (the former receiver) turned into the source, the second became the receiver, and the third became a neutral vertex.

Table 6
Protocol of resource distribution.

t	v_1	v_2	v_3
0	1.000	0.000	0.000
1	0.333	0.333	0.333
2	0.256	0.356	0.389
3	0.250	0.357	0.393
4	0.250	0.357	0.393

From this protocol can be seen that the vector Q^{1*} for given stochastic matrix remains unchanged despite of any changes of capacity matrix and even of the replacement of a receiver.

Let us study the behavior of the network with the growth of total resource.

The increase of resource up to $W = 28$ entails a proportional increase in the coordinates of the limit state. The resource equal to 28 is the threshold value for such a configuration.

Example 7. The matrix R from Example 6; the total resource $W = 28 \approx T$. $Q(0) = (28,\ 0,\ 0)$.

Table 7
Protocol of resource distribution.

t	v_1	v_2	v_3
0	28.000	0.000	0.000
1	22.000	3.000	3.000
2	17.300	5.200	5.500
3	13.653	6.913	7.433
4	10.822	8.243	8.934
5	8.625	9.275	10.100
6	7.169	9.952	10.879
7	7.011	9.997	10.992
8	7.001	10.000	10.999
9	7.000	10.000	11.000

Further growth of the total resource will influence only on the change of the resource in the vertex 2 (the new receiver). Thus, when the total resource is equal to 100, the limit state is: $Q^* = (7.000,\ 82.000,\ 11.000)$.

The threshold value T is derived from the condition that the resource value in the receiver must be equal to its output capacity – in this case, 10, – and all three values of resource in vertices are proportional to the coordinates of the vector Q^{1*}.

Denote the resource values in the vertices corresponding to the value $W = T$, as $\tilde{q}_i$. Then:

$$\tilde{q}_1 = 0.250T;$$
$$\tilde{q}_2 = 0.357T = 10;$$
$$\tilde{q}_3 = 0.393T.$$

From the second ratio: $T = 28$. Then: $\tilde{q}_1 = 7$; $\tilde{q}_3 = 11$.

Now generalize these results.

The vector Q^{1*} and the threshold value T

The vector Q^{1*}

Theorem 1. Any change of the matrix R, invariant with respect to R', remains the vector of unit resource allocation Q^{1*} unchanged.

Proof. Because Q^{1*} is a vector of limit probabilities of the Markov chain, holds: $(R')^{\infty} = \xi Q^{1*}$, where ξ – column vector consisting of n units.

Since the proportional change in rows R does not cause changes in the stochastic matrix R', and, therefore, $(R')^{\infty}$, the vector of limit state Q^{1*} also remains unchanged.

Corollary. Any change of the matrix R, invariant with respect to R', will not effect to the vector Q^*, corresponding to the total resources W $(W \leq T)$ determined by the formula (2).

The threshold value T

Theorem 2. In the connected bidirectional nonsymmetric network with loops which has the only potential attractor with number k:

1) coordinates of the vector $\widetilde{Q} = (\widetilde{q}_1, \widetilde{q}_2, ..., \widetilde{q}_n)$, corresponding to the total resource $W = T$, are as follows:

$$\widetilde{q}_i = \frac{r_k^{out}}{q_k^{1*}} q_i^{1*}; \tag{3}$$

2) the threshold value T is determined as:

$$T = \frac{r_k^{out}}{q_k^{1*}}. \tag{4}$$

Proof. When $W \leq T$ the vector of limit state is proportional to the vector Q^{1*}. When the total resource grows and reaches a value T, one or more vertices in the limit state have resources equal to their output capacity. Since the network has an only potential attractor, then such a vertex is unique and has number k.

Then:

$$\begin{aligned} \widetilde{q}_k &= q_k^{1*} T = r_k^{out}, \\ \widetilde{q}_i &= q_i^{1*} T < r_i^{out}, \quad (i \neq k). \end{aligned} \tag{5}$$

From (5) follows: $T = \frac{r_k^{out}}{q_k^{1*}}$; $\widetilde{q}_i = \frac{r_k^{out}}{q_k^{1*}} q_i^{1*}$.

Remark. The Formula (4) explains Example 5: with a proportional growth of output capacity of the receiver the value T increases in the same proportion.

The attractiveness of vertices and the limit state

The value T is defined by the parameters of the potential attractor. By Theorem 2, the vertex is a potential attractor if with $W = T$ in a limit state it has resource, equal to its output capacity. However, this criterion is not constructive, because a potential attractor and the threshold value are determined through each other.

So it is necessary to formulate a sufficient condition of attractiveness, and the condition of existing of several attractors in a network.

Theorem on attractors. The vertex v_k of nonsymmetric bidirectional network with loops is an attractor if and only if the ratio $\frac{r_i^{out}}{q_i^{1*}}$ specified on the vertices reaches a minimum at $i = k$. The network has more than one attractor if and only if this minimum is reached in more than one vertex.

Proof. For every vertex consider the ratio:

$$T_i = \frac{r_i^{out}}{q_i^{1*}}, = 1, \ldots, n.$$

From (4) follows, that there exists at least one vertex such as: $T_i = T$.

The value T is defined from the condition that by the increase of total resource $W \leq T$ to the value T some vertex first of all will receive the resource equal to its output capacity and will switch to the Rule 1. Thus, to the equality $T_k = T$ was true, for all $i \neq k$ must hold: $q_i^{1*} \cdot T_k \leq r_i^{out}$, or $T_k \leq \frac{r_i^{out}}{q_i^{1*}}$. And this means the condition: $T_k = \frac{r_k^{out}}{q_k^{1*}}$ takes its minimum value.

If this minimum value is reached in several vertices, all of them are potential attractors.

Let us show that if one vertex switches its functioning to the Rule 1, with a further increase of the resource $W > T$ no vertex for which holds $\widetilde{q}_j < r_j^{out}$, will accumulate a resource value to its total output capacity. Moreover, in the limit state other all the vertices for which holds $\frac{r_j^{out}}{q_j^{1*}} > \frac{r_k^{out}}{q_k^{1*}} = T$, always receive the resource $\widetilde{q}_j$, defined by (3).

At $W = T$ the attractor v_k near the limit state receives and sends at every time step $r_k^{out} \pm \varepsilon$ of resource, $\varepsilon \to 0$ as $t \to \infty$.

If the resource value at the vertex v_j exceeds $\widetilde{q}_j$, this vertex will send in every its output arc more than did it at $W = T$, and the attractor will receive more than $r_k^{out} \pm \varepsilon$, but it cannot send more than r_k^{out}. Then the total input exceeds the total output, which contradicts the definition of the limit state.

This implies that with the resource $W > T$ near the limit state the zone Z^+ can contain only those vertices for which holds: $\frac{r_k^{out}}{q_k^{1*}} = T$.

Corollary. If at $W = T$ the vertex in a limit state has the resource less than its output capacity $\widetilde{q}_m < r_m^{out}$, than at any $W > T$ this vertex will have in a limit state the same resource $\widetilde{q}_m$.

The following theorem summarizes the results on any resource value circulating in the network

Theorem on the limit state. The formula for coordinates of vector $Q^* = (q_1^*, q_2^*, ...q_n^*)$ – the limit state of bidirectional connected resource network with loops, – depends on the total resource value W and is defined as follows:

1°. At $W \leq T$, where $T = \min\limits_i \frac{r_i^{out}}{q_i^{1*}}$:

$$q_i^* = q_i^{1*} \cdot W. \tag{6}$$

2°. At $W > T$ the resource value of all non-attractive vertices is:

$$q_i^* = q_i^{1*} \cdot T, \qquad i \neq a_k, \tag{7}$$

where a_k – the numbers of attractors.

a_k are defined from the criterion: $\frac{r_{a_k}^{out}}{q_{a_k}^{1*}}$ – **min**.

The rest resource is distributed among attractors.

Thus, Q^* is determined by three parameters:

1) total output capacities of vertices: r_i^{out};

2) any row of matrix of limit probabilities of Markov chain $(R')^\infty$ and

3) the total resource W.

If $W \leq T$ the limit state is unique. If $W > T$ the vertices are divided into attractors and the set of other vertices. The resource distribution among the attractors in the limit state depends on its initial allocation. All the non-attractive vertices will have the fixed resource value $q_i^* = \widetilde{q}_i = q_i^{1*} \cdot T$.

Conclusion

In this article the collection of networks corresponding to one stochastic matrix was considered. It revealed T the number of dependencies of the limit state on network parameters was revealed. It is shown that the threshold value T and the limit state at $W > T$ depend only on the increase of output capacities of potential attractors. The formulae for the vector of limit state for any total resource W were found; the formula for the threshold value T was found.

From these results follows that if the network has more than one potential attractor, the resource allocation among attractors depends on the initial state. The role of attractors can play not only some receivers, but also neutral vertices without non-positive paths.

Such vertices without non-positive paths form a homogeneous subnetwork within an asymmetric, and behave according to the rules described in [1].

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